

華麗なる三角関数の等式その1

問題 $\alpha + \beta + \gamma = 180^\circ$ のとき、次の等式を証明せよ。

$$(1) \sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

$$(2) \sin \alpha + \sin \beta - \sin \gamma = 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\gamma}{2}$$

$$(3) \cos \alpha + \cos \beta + \cos \gamma = 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} + 1$$

$$(4) \cos \alpha + \cos \beta - \cos \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2} - 1$$

$$(5) \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$$

$$(6) \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin \alpha \sin \beta \sin \gamma$$

$$(7) \cos 2\alpha + \cos 2\beta + \cos 2\gamma = -4 \cos \alpha \cos \beta \cos \gamma - 1$$

$$(8) \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2 \cos \alpha \cos \beta \cos \gamma + 2$$

$$(9) \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = -2 \cos \alpha \cos \beta \cos \gamma + 1$$

$$(10) \tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} = 1$$

華麗なる三角関数の等式その1

問題 $\alpha + \beta + \gamma = 180^\circ$ のとき、次の等式を証明せよ。

$$(1) \sin \alpha + \sin \beta + \sin \gamma = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

解 和を積になおすために、 α 、 β を

$$\alpha = \frac{\alpha + \beta}{2} + \frac{\alpha - \beta}{2}, \quad \beta = \frac{\alpha + \beta}{2} - \frac{\alpha - \beta}{2}$$

と表すと

$$\sin \alpha + \sin \beta = \sin \left(\frac{\alpha + \beta}{2} + \frac{\alpha - \beta}{2} \right) + \sin \left(\frac{\alpha + \beta}{2} - \frac{\alpha - \beta}{2} \right) = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

また $\sin \gamma = \sin(180^\circ - \alpha - \beta) = \sin(\alpha + \beta) = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha + \beta}{2}$ を用いると

$$\begin{aligned} \sin \alpha + \sin \beta + \sin \gamma &= 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} + 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha + \beta}{2} \\ &= 2 \sin \frac{\alpha + \beta}{2} \left(\cos \frac{\alpha - \beta}{2} + \cos \frac{\alpha + \beta}{2} \right) \\ &= 2 \sin \left(90^\circ - \frac{\gamma}{2} \right) \cdot 2 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} = 4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \end{aligned}$$

$$(2) \sin \alpha + \sin \beta - \sin \gamma = 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\gamma}{2}$$

解
$$\begin{aligned} \sin \alpha + \sin \beta - \sin \gamma &= 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} - 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha + \beta}{2} \\ &= 2 \sin \frac{\alpha + \beta}{2} \left(\cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta}{2} \right) \\ &= 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\gamma}{2} \end{aligned}$$

$$(3) \cos \alpha + \cos \beta + \cos \gamma = 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} + 1$$

解 $\cos \gamma = \cos(180^\circ - \alpha - \beta) = -\cos(\alpha + \beta) = -2 \cos^2 \frac{\alpha + \beta}{2} + 1$ を用いると

$$\begin{aligned} \cos \alpha + \cos \beta + \cos \gamma &= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} - 2 \cos^2 \frac{\alpha + \beta}{2} + 1 \\ &= 2 \cos \frac{\alpha + \beta}{2} \left(\cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta}{2} \right) + 1 \\ &= 2 \cos \left(90^\circ - \frac{\gamma}{2} \right) \cdot \sin \frac{\alpha}{2} \sin \frac{\beta}{2} + 1 = 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} + 1 \end{aligned}$$

$$(4) \quad \cos\alpha + \cos\beta - \cos\gamma = 4 \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \sin\frac{\gamma}{2} - 1$$

解 $\cos\alpha + \cos\beta - \cos\gamma = 2 \cos\frac{\alpha+\beta}{2} \cos\frac{\alpha-\beta}{2} + 2 \cos^2\frac{\alpha+\beta}{2} - 1$

$$= 2 \cos\frac{\alpha+\beta}{2} \left(\cos\frac{\alpha-\beta}{2} + \cos\frac{\alpha+\beta}{2} \right) - 1$$

$$= 4 \cos\frac{\alpha}{2} \cos\frac{\beta}{2} \sin\frac{\gamma}{2} - 1$$

$$(5) \quad \tan\alpha + \tan\beta + \tan\gamma = \tan\alpha \tan\beta \tan\gamma$$

解 $\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$, $\tan\gamma = \tan(180^\circ - \alpha - \beta) = -\tan(\alpha+\beta)$ を用いると

$$\tan\alpha + \tan\beta + \tan\gamma = \tan(\alpha+\beta)(1 - \tan\alpha \tan\beta) - \tan(\alpha+\beta)$$

$$= -\tan(\alpha+\beta) \tan\alpha \tan\beta$$

$$= \tan\alpha \tan\beta \tan\gamma$$

$$(6) \quad \sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin\alpha \sin\beta \sin\gamma$$

解 $\sin 2\gamma = \sin(360^\circ - 2\alpha - 2\beta) = -\sin(2\alpha + 2\beta) = -2 \sin(\alpha + \beta) \cos(\alpha + \beta)$ を用いると

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 2 \sin(\alpha + \beta) \cos(\alpha - \beta) - 2 \sin(\alpha + \beta) \cos(\alpha + \beta)$$

$$= 2 \sin(\alpha + \beta) \{ \cos(\alpha - \beta) - \cos(\alpha + \beta) \}$$

$$= 2 \sin(180^\circ - \gamma) \cdot 2 \sin\alpha \sin\beta = 4 \sin\alpha \sin\beta \sin\gamma$$

$$(7) \quad \cos 2\alpha + \cos 2\beta + \cos 2\gamma = -4 \cos\alpha \cos\beta \cos\gamma - 1$$

解 $\cos 2\gamma = \cos(360^\circ - 2\alpha - 2\beta) = \cos(2\alpha + 2\beta) = 2 \cos^2(\alpha + \beta) - 1$ を用いると

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma = 2 \cos(\alpha + \beta) \cos(\alpha - \beta) + 2 \cos^2(\alpha + \beta) - 1$$

$$= 2 \cos(\alpha + \beta) \{ \cos(\alpha - \beta) + \cos(\alpha + \beta) \} - 1$$

$$= 2 \cos(180^\circ - \gamma) \cdot 2 \cos\alpha \cos\beta - 1 = -4 \cos\alpha \cos\beta \cos\gamma - 1$$

$$(8) \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2 \cos\alpha \cos\beta \cos\gamma + 2$$

解 (7) を用いる。

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{1 - \cos 2\alpha}{2} + \frac{1 - \cos 2\beta}{2} + \frac{1 - \cos 2\gamma}{2}$$

$$= \frac{3}{2} - \frac{1}{2}(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = \frac{3}{2} - \frac{1}{2}(-4 \cos\alpha \cos\beta \cos\gamma - 1)$$

$$= 2 \cos\alpha \cos\beta \cos\gamma + 2$$

$$(9) \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma = -2 \cos\alpha \cos\beta \cos\gamma + 1$$

$$\begin{aligned} \text{解} \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \frac{1 + \cos 2\alpha}{2} + \frac{1 + \cos 2\beta}{2} + \frac{1 + \cos 2\gamma}{2} \\ &= \frac{3}{2} + \frac{1}{2}(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = \frac{3}{2} + \frac{1}{2}(-4 \cos\alpha \cos\beta \cos\gamma - 1) \\ &= -2 \cos\alpha \cos\beta \cos\gamma + 1 \end{aligned}$$

$$(10) \quad \tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} = 1$$

$$\text{解} \quad \tan \frac{\gamma}{2} = \tan \left(90^\circ - \frac{\alpha + \beta}{2} \right) = \frac{1}{\tan \frac{\alpha + \beta}{2}}, \quad \tan \frac{\alpha + \beta}{2} = \frac{\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}}{1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}} \quad \text{を用いると}$$

$$\begin{aligned} &\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} \\ &= \tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \left(\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} \right) \tan \frac{\gamma}{2} \\ &= \tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \left(1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \right) \tan \frac{\alpha + \beta}{2} \cdot \frac{1}{\tan \frac{\alpha + \beta}{2}} = 1 \end{aligned}$$