

## 華麗なる三角関数の等式その2

問題  $\alpha + \beta + \gamma + \delta = 360^\circ$  のとき、次の等式を証明せよ。

$$(1) \cos \alpha + \cos \beta + \cos \gamma + \cos \delta = 4 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2}$$

$$(2) \cos \alpha - \cos \beta + \cos \gamma - \cos \delta = 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2}$$

$$(3) \sin \alpha + \sin \beta + \sin \gamma + \sin \delta = 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2}$$

$$(4) \sin \alpha - \sin \beta + \sin \gamma - \sin \delta = 4 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2}$$

$$(5) \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \cos \frac{\delta}{2} = \sin \frac{\alpha}{2} \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \sin \frac{\delta}{2}$$

$$(6) \sin \frac{\alpha}{2} \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \sin \frac{\delta}{2} = \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2}$$

注 :  $\sin \frac{\beta + \gamma}{2}$  は  $\sin \frac{\alpha + \delta}{2}$  に、 $\sin \frac{\gamma + \alpha}{2}$  は  $\sin \frac{\beta + \delta}{2}$  に置き換えてもよい。

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問題  $\alpha + \beta + \gamma + \delta = 360^\circ$  のとき、次の等式を証明せよ。

$$(1) \quad \cos \alpha + \cos \beta + \cos \gamma + \cos \delta = 4 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2}$$

解  $\cos \frac{\beta + \delta}{2} = \cos \left( 180^\circ - \frac{\alpha + \gamma}{2} \right) = -\cos \frac{\alpha + \gamma}{2}$  および

$$\frac{\beta - \delta}{2} = \frac{\alpha + 2\beta + \gamma - 360^\circ}{2} = \frac{\alpha + \beta}{2} + \frac{\beta + \gamma}{2} - 180^\circ \text{ より}$$

$$\cos \frac{\beta - \delta}{2} = -\cos \left( \frac{\alpha + \beta}{2} + \frac{\beta + \gamma}{2} \right) \text{ を用いると}$$

$$\cos \alpha + \cos \beta + \cos \gamma + \cos \delta = 2 \cos \frac{\alpha + \gamma}{2} \cos \frac{\alpha - \gamma}{2} + 2 \cos \frac{\beta + \delta}{2} \cos \frac{\beta - \delta}{2}$$

$$= 2 \cos \frac{\alpha + \gamma}{2} \cos \left( \frac{\alpha + \beta}{2} - \frac{\beta + \gamma}{2} \right) + 2 \cos \frac{\alpha + \gamma}{2} \cos \left( \frac{\alpha + \beta}{2} + \frac{\beta + \gamma}{2} \right)$$

$$= 2 \cos \frac{\alpha + \gamma}{2} \cdot 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2}$$

$$= 4 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2}$$

$$(2) \quad \cos \alpha - \cos \beta + \cos \gamma - \cos \delta = 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2}$$

解  $\cos \alpha - \cos \beta + \cos \gamma - \cos \delta = 2 \cos \frac{\alpha + \gamma}{2} \cos \frac{\alpha - \gamma}{2} - 2 \cos \frac{\beta + \delta}{2} \cos \frac{\beta - \delta}{2}$

$$= 2 \cos \frac{\alpha + \gamma}{2} \cos \left( \frac{\alpha + \beta}{2} - \frac{\beta + \gamma}{2} \right) - 2 \cos \frac{\alpha + \gamma}{2} \cos \left( \frac{\alpha + \beta}{2} + \frac{\beta + \gamma}{2} \right)$$

$$= 2 \cos \frac{\alpha + \gamma}{2} \cdot 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} = 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \cos \frac{\gamma + \alpha}{2}$$

$$(3) \quad \sin \alpha + \sin \beta + \sin \gamma + \sin \delta = 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2}$$

解  $\sin \frac{\beta + \delta}{2} = \sin \left( 180^\circ - \frac{\alpha + \gamma}{2} \right) = \sin \frac{\alpha + \gamma}{2}$  を用いると

$$\sin \alpha + \sin \beta + \sin \gamma + \sin \delta = 2 \sin \frac{\alpha + \gamma}{2} \cos \frac{\alpha - \gamma}{2} + 2 \sin \frac{\beta + \delta}{2} \cos \frac{\beta - \delta}{2}$$

$$= 2 \sin \frac{\alpha + \gamma}{2} \cos \left( \frac{\alpha + \beta}{2} - \frac{\beta + \gamma}{2} \right) + 2 \sin \frac{\alpha + \gamma}{2} \cos \left( \frac{\alpha + \beta}{2} + \frac{\beta + \gamma}{2} \right)$$

$$= 2 \sin \frac{\alpha + \gamma}{2} \cdot 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2}$$

$$= 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2}$$

$$(4) \sin \alpha - \sin \beta + \sin \gamma - \sin \delta = 4 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2}$$

$$\begin{aligned} \text{解} \quad \sin \alpha - \sin \beta + \sin \gamma - \sin \delta &= 2 \sin \frac{\alpha + \gamma}{2} \cos \frac{\alpha - \gamma}{2} - 2 \sin \frac{\beta + \delta}{2} \cos \frac{\beta - \delta}{2} \\ &= 2 \sin \frac{\alpha + \gamma}{2} \cos \left( \frac{\alpha + \beta}{2} - \frac{\beta + \gamma}{2} \right) + 2 \sin \frac{\alpha + \gamma}{2} \cos \left( \frac{\alpha + \beta}{2} + \frac{\beta + \gamma}{2} \right) \\ &= 2 \sin \frac{\alpha + \gamma}{2} \cdot 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2} \\ &= 4 \cos \frac{\alpha + \beta}{2} \cos \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2} \end{aligned}$$

$$(5) \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \cos \frac{\delta}{2} = \sin \frac{\alpha}{2} \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \sin \frac{\delta}{2}$$

$$\begin{aligned} \text{解} \quad \left( \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \cos \frac{\delta}{2} \right) - \left( \sin \frac{\alpha}{2} \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \sin \frac{\delta}{2} \right) \\ &= \cos \frac{\alpha + \beta}{2} + \cos \frac{\gamma + \delta}{2} \\ &= \cos \frac{\alpha + \beta}{2} + \cos \left( 180^\circ - \frac{\alpha + \beta}{2} \right) = 0 \end{aligned}$$

$$\text{ゆえに} \quad \cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \cos \frac{\gamma}{2} \cos \frac{\delta}{2} = \sin \frac{\alpha}{2} \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \sin \frac{\delta}{2}$$

$$(6) \sin \frac{\alpha}{2} \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \sin \frac{\delta}{2} = \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2}$$

$$\begin{aligned} \text{解} \quad \sin \frac{\alpha}{2} \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \sin \frac{\delta}{2} &= \sin \left( \frac{\gamma + \alpha}{2} - \frac{\gamma}{2} \right) \sin \frac{\beta}{2} + \sin \frac{\gamma}{2} \sin \frac{\alpha + \beta + \gamma}{2} \\ &= \sin \frac{\beta}{2} \left( \sin \frac{\gamma + \alpha}{2} \cos \frac{\gamma}{2} - \cos \frac{\gamma + \alpha}{2} \sin \frac{\gamma}{2} \right) \\ &\quad + \left( \sin \frac{\gamma + \alpha}{2} \cos \frac{\beta}{2} + \cos \frac{\gamma + \alpha}{2} \sin \frac{\beta}{2} \right) \sin \frac{\gamma}{2} \\ &= \left( \sin \frac{\beta}{2} \cos \frac{\gamma}{2} + \cos \frac{\beta}{2} \sin \frac{\gamma}{2} \right) \sin \frac{\gamma + \alpha}{2} \\ &= \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2} \end{aligned}$$

**別解** 直径1の円に内接する四角形 ABCD について右図のように角をとり、正弦定理とトレミーの定理  $AB \cdot CD + BC \cdot DA = AC \cdot BD$  を用いる。

